Some instances of a sub-permutation problem on pattern avoiding permutations

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Abstract

We study here the enumeration problem of permutations which satisfy certain additional constraints. Given a class of permutations $\mathcal{K}$ a pattern μ and a fixed integer j, we ask for the number of permutations avoiding μ whose biggest *sub-permutation* in $\mathcal{K}$ has size bounded by j. We provide several new results considering different instances of this problem depending on μ and $\mathcal{K}$. In particular, we derive enumerations when the avoided pattern μ is 312, 123 and 1-32 and when the considered test sets $\mathcal{K}$ are also of pattern avoidance type. Most (but not all) of the cases studied correspond to interesting sub-tree properties of binary trees. In this sense, by the use of pattern avoiding conditions, we extend a problem considered by Flajolet et al. in a previous work.

1 Introduction

The aim of this paper is to provide several enumerative results studying *sub-permutations*. Sub-permutations, as defined in Section 2, correspond to the classical concept of sub-tree according to a well known bijection which maps the set of pemutations of size n onto the set of binary *increasing* trees having n nodes.

The main section of the paper, i.e. Section 3, is dedicated to solve some instances of the following general problem.

Problem: *given a class of permutations $\mathcal{K}$, what is the number of permutations π whose biggest sub-permutation in $\mathcal{K}$ has size bounded by a fixed number?*

If the permutation π does not have to satisfy any pattern avoiding condition the problem has been addressed by Flajolet, Gourdon and

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Martinez [5]. Here, we introduce the condition of pattern avoidance and derive several novel results when the avoided pattern (μ) is of length three. More precisely, we consider $\mu_1 = 312$, $\mu_2 = 123$ and $\mu_3 = 1-32$ where the first two are classical Catalan patterns and the third is a generalized pattern according to [1].

In Section 3.1 we consider four possible instances of class $\mathcal{K}$. For μ_1 we consider $\mathcal{K} = Av(213)$ and $\mathcal{K}$ equal to the set of odd alternating permutations. In particular we prove that, for $n \to \infty$, taking at random a permutation in $\pi \in Av_n(\mu_1)$ the expected value of the size of the biggest sub-permutation of π belonging to $Av(213)$ goes like $\log_2(n)$. For μ_2 we consider $\mathcal{K} = Av(21)$ and $\mathcal{K} = Av(12)$ showing how, in the last case, the number of permutations of $Av_n(\mu_2)$ whose biggest sub-permutation in $\mathcal{K}$ has size bounded by a fixed number j is related to the number of *Dyck* paths of size n avoiding the pattern $U^{j+2}D$.

In Section 3.2 we focus on permutations avoiding μ_3 characterizing them in terms of paths of the associated trees. We then consider $\mathcal{K} = Av(12)$ and provide a limit approach to the enumeration of $Av_n(\mu_3)$ which is given by *Bell* numbers.

In Section 4 we introduce the concept of pattern avoidance in terms of sub-permutations. We find the probability to detect a pattern 213 in a permutation looking just at the sub-permutation generated by the entry 2 and we generalize our result considering any pattern σ. We think this new kind of pattern related problems deserves further studies. Indeed it seems intersting to know in which cases one can predict properties of a permutation when it is allowed to inspect only a limited part of it.

Finally, we note that the kind of questions formulated here in the context of permutations are generic for all combinatorial objects for which a notion of *sub-structure* is defined.

2 Preliminaries

The set of permutations of size n is denoted by $\mathcal{S}_n$ and $\mathcal{S} = \bigcup_n \mathcal{S}_n$. We assume the reader to be familiar with the classical concept of pattern avoidance in permutations. The subset of $\mathcal{S}_n$ made of those permutations avoiding the pattern σ is usually denoted by $Av_n(\sigma)$ and $Av(\sigma) = \bigcup_n (Av_n(\sigma))$.

Let $\pi = (\pi_1 \pi_2 \ldots \pi_n)$ be a permutation. For a given entry π_i we define $s_\pi(\pi_i)$ as the biggest sub-string [1] of π which contains π_i and whose entries are greater than or equal to π_i. Furthermore let $g_\pi(\pi_i)$ be the permutation obtained rescaling $s_\pi(\pi_i)$. We call $g_\pi(\pi_i)$ the *sub-permutation* of π generated by π_i. The set of sub-permutations

[1] Sub-strings are made of adjacent elements of π.

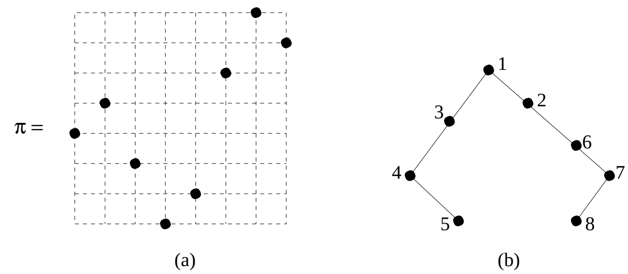

Figure 1: (a) The permutation $\pi = (45312687)$; (b) the tree associated with π by ϕ.

$(g_\pi(\pi_i))_{i=1\dots n}$ is denoted by G_π. As an example consider the permutation π which is depicted in Fig. 1 (a). In this case G_π is made of $g_\pi(5) = g_\pi(8) = (1)$, $g_\pi(4) = (12)$, $g_\pi(7) = (21)$, $g_\pi(3) = (231)$, $g_\pi(6) = (132)$, $g_\pi(2) = (1243)$ and $g_\pi(1) = (45312687)$.

Given a class of permutations $\mathcal{K}$ we denote by $\gamma_\pi(\mathcal{K})$ the size of the biggest sub-permutation of π belonging to $\mathcal{K}$.

The concept of sub-permutations is related to the one of sub-trees. To illustrate this correspondance we make use of a well known bijection between the set $\mathcal{S}_n$ and the set of binary increasing trees of size n, denoted by $\mathcal{T}_n$. We recall that a planar rooted tree t, having n nodes, belongs to $\mathcal{T}_n$ when:

- each node has outdegree $0, 1$ or 2. Nodes of outdegree 0 are called *leaves*;
- each node (except for the root) can be left or right oriented with respect to its direct ancestor (see Fig. 1 (b));
- each node is labelled (bijectively) with a number in $\{1, ..., n\}$ in such a way going from the root of t to any leaf of t we find an increasing sequence of numbers.

The bijection $\phi : \mathcal{T}_n \to \mathcal{S}_n$ is given by the following procedure:

i) given a tree t each leaf of t collapses into its direct ancestor whose label is then modified receiving on the left the label of the left child (if any) and on the right the label of the right child (if any). We obtain in this way a new tree whose nodes are labelled with sequences of numbers;

ii) starting from the obtained tree go to step *i*).

The algorithm ϕ ends when the tree t is reduced to a single node whose label is then a permutation $\phi(t)$ of size n. For an example see Fig. 1.

The link between sub-permutations and sub-trees is expressed by the following proposition.

Proposition 1 *Given a permutation $\pi = \phi(t)$, let t_i be the (re-scaled) sub-tree of t generated by the node π_i, then the sub-permutation $g_\pi(\pi_i)$ is equal to $\phi(t_i)$.*

3 On the biggest sub-permutation of $\pi \in Av(\mu)$ belonging to $\mathcal{K}$

In this section we want to determine the number of permutations in $Av_n(\mu)$ with the condition that the size of the biggest sub-permutation in $\mathcal{K}$ is bounded by a fixed number. We consider several instances of μ and $\mathcal{K}$ and find new enumerative results. Some of them are connected to already known combinatorial problems.

3.1 Two classical patterns of length three

In this section we study the patterns $\mu = 312$ and $\mu = 123$. It is well known that both the classes $Av(312)$ and $Av(123)$ are enumerated by *Catalan* numbers whose first terms are

$$c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14, c_5 = 42, c_6 = 132, c_7 = 429.$$

The associated ordinary generating function is [6]

$$C(x) = \sum_{n\geq 0} c_n x^n = \frac{1-\sqrt{1-4x}}{2x}, \tag{1}$$

which has a closed form for its n-th coefficient given by

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Furthermore, it is well known that

$$c_n \sim \frac{4^n}{\sqrt{\pi n^3}},$$

if n is large.

3.1.1 $\mu = 312$ and binary rooted planar trees

We denote by $\mathcal{B}_n$ the set of binary rooted planar trees with n internal nodes, where each internal node has outdegree two. It is well known that one can bijectively map the set $\mathcal{B}_n$ onto the set $Av_n(312)$. In

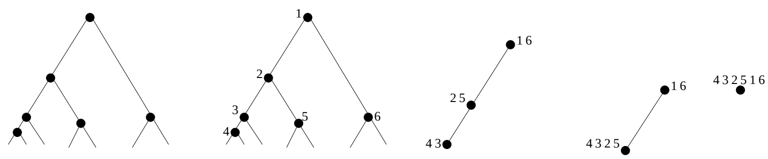

Figure 2: The mapping ψ.

particular we use a bijection $\psi : \mathcal{B}_n \to Av_n(312)$ which works similarly to the mapping ϕ of Section 2, as follows.

Take $t \in \mathcal{B}_n$ and visit its nodes according to the pre-order traversal labelling each node of outdegree two in increasing order starting with the label 1 for the root. After this first step one has a tree labelled with integers at its nodes of outdegree two. Each leaf now collapses to its direct ancestor which takes a new label receiving on the left (resp. right) the label of its left (resp. right) child. We go on collapsing leaves until we achieve a tree made of one node which is labelled with a permutation of size n. See Fig. 2 for an instance of this mapping.

$\mathcal{K} = Av(213)$ and caterpillar sub-trees

Let now $\mathcal{K} = Av(213)$. Through the bijection ψ we see that, given a permutation $\pi \in Av(312)$, the parameter $\gamma_\pi(\mathcal{K})$ corresponds then to the size (number of internal nodes) of the biggest sub-tree of $\psi^{-1}(\pi)$ which is a caterpillar: a tree t in $\mathcal{B}_n$ is a *caterpillar* of size n, if each node of t is a leaf or has at least one leaf as a direct descendant.

If we define P_j as the *ordinary* generating function

$$P_j(x) = \sum_{n\geq 0} v_{j,n} x^n$$

counting those permutations π in $Av_n(\mu)$ having $\gamma_\pi(\mathcal{K}) \leq j$, we observe that P_j must then satisfy the equation

$$P_j = 1 + x{P_j}^2 - 2^j x^{j+1}. \tag{2}$$

Indeed, a tree t whose biggest caterpillar sub-tree is of size at most j, is either a leaf or it is built by appending to the root of t two trees t_1 and t_2 of the same class. We must exclude the case in which one of the two, t_1 or t_2, has size 0, i.e., is a leaf, and the other one is a caterpillar of size j. Since there are exactly 2^{j-1} caterpillars of size j the previous formula follows.

From (2) we obtain

$$P_j(x) = \frac{1 - \sqrt{1 - 4x + 2^{j+2}x^{j+2}}}{2x} \tag{3}$$

which gives an expansion, for $j = 1$, with the following coefficients

$$1, 1, 0, 1, 2, 6, 16, 45, 126, 358, 1024$$

which correspond to sequence $A025266$ of [9]. As stated there, they also count classes of Motzkin paths satisfying some constraints.

The asymptotic behaviour of the coefficients of P_j follows by standard analytic methods [6]. When n becomes large the coefficients grow like

$$v_{j,n} \sim \frac{1}{4}\sqrt{\frac{4\rho_j - (j+2)2^{j+2}\rho_j^{j+2}}{\pi n^3}} \left(\frac{1}{\rho_j}\right)^{n+1}, \tag{4}$$

where ρ_j is the smallest positive solution of

$$1 - 4x + 2^{j+2}x^{j+2} = 0.$$

Starting from the following inequality

$$\frac{1}{4} < \rho_j < \frac{2}{5},$$

which is strightforward to prove for all $j \geq 1$, and applying the so-called *bootstrapping* technique [6] one can show that ρ_j tends to $1/4$ as

$$\rho_j = \frac{1}{4} + \frac{1}{2^{j+4}} + O\left(j\left(\frac{2}{5}\right)^j\right).$$

The main result we want to present in this section concerns the average of $\gamma = \gamma_\pi(\mathcal{K})$ when the size of the permutation π is large.

Proposition 2 *When $\pi \in Av_n(\mu)$, the expected value of $\gamma_\pi(\mathcal{K})$ is asymptotically equivalent to* $\log_2(n)$.

Proof. If $n \geq 1$ we can express the desired average value as

$$\begin{aligned}
E_n(\gamma) &= \frac{1v_{1,n} + \sum_{j\geq 1}(j+1)(v_{j+1,n} - v_{j,n})}{c_n} \\
&= \frac{-v_{1,n} - \ldots - v_{n-1,n} + nv_{n,n} + \sum_{j\geq n}(j+1)(v_{j+1,n} - v_{j,n})}{c_n} \\
&= \frac{-v_{1,n} - \ldots - v_{n-1,n} + nc_n + \sum_{j\geq n}(c_n - v_{j,n})}{c_n} \\
&= \frac{\sum_{j=1}^{n-1}(c_n - v_{j,n}) + c_n + \sum_{j\geq n}(c_n - v_{j,n})}{c_n} \\
&= \frac{c_n + \sum_{j\geq 1}(c_n - v_{j,n})}{c_n} \\
&= 1 + \frac{\sum_{j\geq 1}(c_n - v_{j,n})}{c_n}
\end{aligned}$$

In the previous calculation we have used the fact that for $j \geq n$ we always have $v_{j,n} = c_n$.

It is sufficient now to find the n-th term of the function

$$U(x) = \sum_{j \geq 1} (C(x) - P_j(x)) = \frac{\sqrt{1-4x}}{2x} \sum_{j \geq 1} \left(\sqrt{1 + \frac{2^{j+2}x^{j+2}}{1-4x}} - 1 \right).$$

In what follows we want to find a function $\tilde{U}$ which estimates U near the dominant singularity $1/4$. According to [6], when n is large, the n-th term of the Taylor expansion of $\tilde{U}$ provides an approximation of $[x^n]U(x)$. Our approach results to be similar to the one used in Section 3 of [7].

Let us fix x near $1/4$ and let us consider the threshold function

$$j_0 = \log_2 \frac{1}{|1-4x|}.$$

Then, supposing $j \geq j_0 - 1$, we have that

$$\sqrt{1 + \frac{2^{j+2}x^{j+2}}{1-4x}} \sim \sqrt{1 + \frac{1}{2^{j+2}(1-4x)}} \sim 1 + \frac{1}{2^{j+3}(1-4x)},$$

while if $j < j_0 - 1$ we can use the approximation

$$\sqrt{1 + \frac{2^{j+2}x^{j+2}}{1-4x}} \sim \sqrt{1 + \frac{1}{2^{j+2}(1-4x)}} \sim \sqrt{\frac{1}{2^{j+2}(1-4x)}}.$$

For x sufficiently close to $1/4$ we estimate $U(x)$ as

$$
\begin{aligned}
U(x) &\sim \frac{\sqrt{1-4x}}{2x\sqrt{1-4x}}\sum_{j\geq 1}^{j_0-2}\sqrt{\frac{1}{2^{j+2}}} - \frac{\sqrt{1-4x}}{2x}\sum_{j\geq 1}^{j_0-2} 1 \\
&\quad + \frac{\sqrt{1-4x}}{2x(1-4x)}\sum_{j\geq j_0-1}\frac{1}{2^{j+3}} \\
&= \frac{1}{4x}\sum_{j\geq 1}^{j_0-2}\sqrt{\frac{1}{2^j}} - \frac{\sqrt{1-4x}}{2x}(j_0-2) + \frac{1}{16x\sqrt{1-4x}}\sum_{j\geq j_0-1}\frac{1}{2^j} \\
&= \frac{1}{4x}\frac{-\sqrt{2}+2\sqrt{2}\sqrt{2^{-j_0}}}{-2+\sqrt{2}} \\
&\quad - \frac{\sqrt{1-4x}}{2x}\left(\log_2\left(\frac{1}{|1-4x|}\right)-2\right) + \frac{2^{2-j_0}}{16x\sqrt{1-4x}} \\
&= \frac{\log(2)}{x\log(16)} + \frac{\sqrt{2}\log(2)}{x\log(16)} - \frac{2\sqrt{2-8x}\log(2)}{x\log(16)} \\
&\quad - \frac{\sqrt{1-4x}\log(2)}{x\log(16)} - \frac{2\sqrt{1-4x}\log\left(\frac{1}{4-16x}\right)}{x\log(16)} \\
&\sim -\frac{2\sqrt{1-4x}\log\left(\frac{1}{1-4x}\right)}{x\log(16)}.
\end{aligned}
$$

Using the previous calculation we can say that

$$\tilde{U}(x) = -\frac{2\sqrt{1-4x}\log\left(\frac{1}{1-4x}\right)}{x\log(16)} \tag{5}$$

approximates $U(x)$ near its dominant singularity $1/4$. It follows that when $n \to \infty$

$$E_n(\gamma) \sim \frac{[x^n]\tilde{U}(x)}{c_n}.$$

Applying standard methods [6, 7] to (5) we find that

$$[x^n]\tilde{U}(x) \sim \frac{4^{n+1}\log(n)}{\log(16)\sqrt{\pi n^3}}.$$

Dividing by the asymptotic behaviour of Catalan numbers gives the claim. □

As a test one can consider the following table where, for several values of n, we compare the true $E_n(\gamma)$ with the approximation given by Proposition 2.

n	10	20	50	100	200	500	1000
$E_n(\gamma)$	3.596	4.172	5.227	6.121	7.058	8.336	9.319
$\log_2(n)$	3.321	4.321	5.643	6.643	7.643	8.965	9.965

To conclude this section it is interesting to observe another possible application of the function P_j described in (3). Indeed the number of permutations in $Av_n(\mu)$ having no sub-permutation of size j in $\mathcal{K}$ is given by the n-th coefficient of

$$P_{j-1}(x) = \frac{1 - \sqrt{1 - 4x + 2^{j+1}x^{j+1}}}{2x}. \tag{6}$$

We will compare this result with the analogous one provided in the next section.

When $\mathcal{K}$ is the set of (odd) alternating permutations

A tree in $\mathcal{B}_n$, where n is odd, is called *strictly binary* if, removing the leaves, the remaining nodes have either out-degree 0 or out-degree 2. The corresponding (through the mapping ψ) sub-set of $Av_n(\mu)$ consists of permutations $\pi = (\pi_1\pi_2...\pi_n)$ characterized by the following property: either $n = 0, 1$ or $\pi_1 > \pi_2 < \pi_3 > ... < \pi_n$. It is well known that the number of strictly binary trees of size $2m + 1$ is c_m.

The parameter $\gamma_\pi(\mathcal{K})$ corresponds in this case to the size of the biggest strictly binary sub-tree of $\psi^{-1}(\pi)$ and, obviously, the only possible values of $\gamma_\pi(\mathcal{K})$ are odd. The ordinary generating function $L_j(x)$, with $j = 2m + 1$, counts those trees in $\mathcal{B}_n$ having *at least* one strictly binary sub-tree of size j. Equivalently, it can be seen as the function counting the permutations π in $Av_n(\mu)$ with $\gamma_\pi(\mathcal{K}) \geq j$. L_j must then satisfy

$$L_j = c_m x^j + x{L_j}^2 + 2xL_j(C - L_j), \tag{7}$$

where C is the generating function of Catalan numbers as in (1).

Indeed a tree counted by L_j is either a strictly binary tree of size j (first summand) or it is built by appending to the root two trees, where at least one of them contains a strictly binary sub-tree of size j (second and third summand).

Solving (7) we obtain

$$L_j(x) = \frac{\sqrt{1 - 4x + 4c_m x^{j+1}} - \sqrt{1 - 4x}}{2x}. \tag{8}$$

From this, we can also determine the number of those permutations avoiding the pattern 312 and without (odd) alternating sub-permutation of size j. This is given by

$$C(x) - L_j(x) = \frac{1 - \sqrt{1 - 4x + 4c_m x^{j+1}}}{2x}. \tag{9}$$

We now want to compare results (6) and (9) for a fixed $j = 2m+1$ with m large. Let us consider $a_m > 1/4$ (resp. $b_m > 1/4$) defined as the smallest positive root of $1 - 4x + 4c_m x^{2m+2} = 0$ (resp. $1 - 4x + 2^{2m+2}x^{2m+2} = 0$). By asymptotic considerations, we know that when $m \to \infty$ both a_m and b_m tend to $1/4$. But we can be more precise: when m is large, the equality

$$1 - 4a_m + 4c_m a_m^{2m+2} = 0 = 1 - 4b_m + 2^{2m+2} b_m^{2m+2}$$

implies

$$\frac{a_m}{b_m} = \frac{-1 + 4^m b_m^{2m+1}}{-1 + c_m a_m^{2m+1}} \sim \frac{-1 + (\frac{1}{4})^{m+1}}{-1 + c_m (\frac{1}{4})^{2m+1}} < 1.$$

For example, if $m = 5$, we have that $a_m/b_m = 0.999765$ while the previous approximation gives 0.999766.

By standard methods (see [6]) one can compute the asymptotic behaviour of the n-th coefficient of (9) as

$$\frac{1}{4}\sqrt{\frac{4a_m - (2m+2)4c_m a_m^{2m+2}}{\pi n^3}} \left(\frac{1}{a_m}\right)^{n+1}$$

while the behaviour of the n-th coefficient of (6) is given by (4) (considering $b_m = \rho_{j-1}$) as

$$\frac{1}{4}\sqrt{\frac{4b_m - (2m+2)2^{2m+2} b_m^{2m+2}}{\pi n^3}} \left(\frac{1}{b_m}\right)^{n+1}.$$

From these results follows

Proposition 3 *For a fixed and sufficiently large m, when $n \to \infty$, the ratio*

$$\frac{|\{\pi \in Av_n(\mu) : (G_\pi \cap Av(213) \cap \mathcal{S}_{2m+1}) = \emptyset\}|}{|\{\pi \in Av_n(\mu) : (G_\pi \cap \mathcal{K} \cap \mathcal{S}_{2m+1}) = \emptyset\}|} \tag{10}$$

goes to 0 *equally fast as*

$$k_m \left(\frac{-1 + (\frac{1}{4})^{m+1}}{-1 + c_m (\frac{1}{4})^{2m+1}}\right)^{n+1}, \tag{11}$$

where k_m is a constant depending only on m.

In the following table we compare the values of the ratio of Proposition 3 with the asymptotic ratio (11) for $m = 5$ and different values of n.

n	50	500	1000	5000	10000
ratio (10)	0.986	0.887	0.789	0.308	0.095
$k_m = 1$ in (11)	0.988	0.889	0.791	0.310	0.096

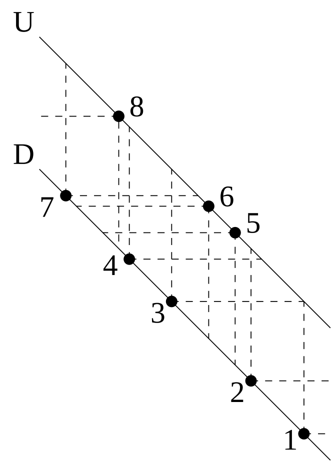

Figure 3: The two-line representation of $(78436521) \in Av_8(123)$; for example 6 covers 4 and 3

3.1.2 $\mu = 123$, a generating tree approach

In what follows let $\mu = 123$ be fixed. It is a well known fact that the entries of a permutation $\pi \in Av(\mu)$ can be seen as points lying on two non-intersecting lines as depicted in Fig. 3. Viceversa, each permutation which can be drawn in such a way avoids μ. A point p_1 on the line D is *covered* by a point p_2 of the upper line U if p_2 is on the right and above p_1. In order to avoid redundancies in the two-line representation of a permutation we have to respect the following rule: a point belongs to the upper line U if and only if it covers at least one point of D.

It is useful to observe that the set $Av_{n+1}(\mu)$ can be generated by the permutations of $Av_n(\mu)$ adding the rightmost entry. Taken $\pi \in Av_n(\mu)$, let us define $u(\pi)$ as the right-most point placed on the line U (if any) and let $l(\pi)$ be the number of elements which are placed on D with a smaller ordinate than $u(\pi)$. In order to create a permutation π' of size $n+1$ we add on the right of π a new element p'. If p' is placed on the D line then $l(\pi') = l(\pi) + 1$. Otherwise p' can be placed on the U line in exactly $l(\pi)$ different ways, see Fig. 4. In this case $l(\pi')$ ranges between 1 and $l(\pi)$.

In conclusion, we have described a *generating tree* procedure which starts with the single element permutation and, always adding the rightmost entry, generates all the permutations in $Av(123)$. Furthermore, if we consider $(l) = (l(\pi))$ the previous procedure can be summarized as

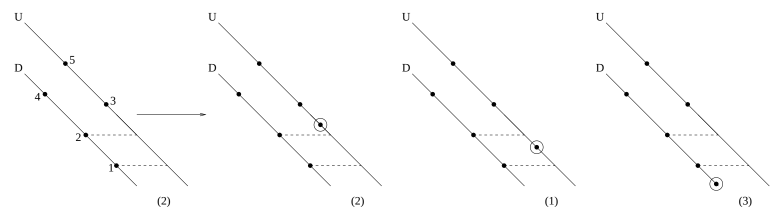

Figure 4: The permutations produced starting from (45231). The new rightmost entry is highlighted.

$$(l) \rightsquigarrow (1), (2), ..., (l+1). \tag{12}$$

This rule can be viewed as a tree generating procedure, where a label (l) is a node with $(l+1)$ descendants. If the starting label (i.e. the root of the tree) is (0), the number of nodes at level n is given by c_n. More details on generating trees can be found in [2].

$\mathcal{K} = Av(21)$, the biggest increasing sub-permutation

In this section $\gamma_\pi(\mathcal{K}) \in \{1,2\}$ is the biggest increasing sub-permutation of a given $\pi \in Av(\mu)$ and we notice that $\gamma_\pi(\mathcal{K}) = 2$ if and only if the first (from left to right) three entries of π are in the order 231. Indeed, if $a < b$ are the entries of an increasing sub-permutation of size 2, such a sub-permutation is the one generated by a. It follows that a must be the leftmost entry of π while b the topmost. In order to 'close' the sub-permutation, we also need the presence of a point $c < a$ placed immediately on the right of b.

To count the permutations $\pi \in Av(\mu)$ of size $n(\pi) \geq 3$ with the biggest increasing sub-permutation of size 2 we have just to consider rule (12) with a starting point given by $(2) = l(231)$.

By standard computations, see again [2], one finds that the ordinary generating function

$$M_2 = M_2(x,y) = \sum_\pi x^{n(\pi)} y^{l(\pi)}$$

satisfies the functional equation

$$\left(1 + \frac{xy^2}{1-y}\right) M_2 = x^3y^2 + \frac{xy}{1-y} M_2(x,1) \tag{13}$$

which can be solved by the *kernel method* finding

$$M_2(x,1) = \frac{1}{4}\left(-1+\sqrt{1-4x}\right)\left(-1+\sqrt{1-4x}+2x\right).$$

A closed formula for the coefficients is given by

$$[x^n]M(x,1) = \frac{3(2n-4)!}{(n-3)!n!} \text{ (with } n \geq 3).$$

In conclusion we can state that

Proposition 4 *If $n \geq 3$, the number of permutations in $Av_n(123)$ having an increasing sub-permutation of size 2 is*

$$a_n = \frac{3(2n-4))!}{(n-3)!n!}.$$

Furthermore, the ones having the biggest increasing sub-permutation of size 1 are (with $n \geq 3$)

$$b_n = c_n - a_n,$$

where c_n is the n-th Catalan number.

For $n = 3, ..., 10$ the following table shows the values of a_n and b_n.

$n =$	3	4	5	6	7	8	9	10
a_n	1	3	9	28	90	297	1001	3432
b_n	4	11	33	104	339	1133	3861	13364

The coefficients a_n are the (shifted) entries of sequence $A000245$ of [9], while the numbers b_n do not appear there.

$\mathcal{K} = Av(12)$ and pattern avoidance in Dyck paths

In this section $\gamma_\pi(\mathcal{K})$ is the biggest decreasing sub-permutation of a given $\pi \in Av(\mu)$. Looking at the two line representation of π, which is described in Section 3.1.2, we see that $g_\pi(\pi_i)$ is decreasing if and only if it corresponds to a sequence of adjacent points (adjacent with respect to their abscissas) lying on the line U, with the only exception of $\pi \in Av(12)$. For example, the permutation depicted in Fig. 3 has three decreasing sub-permutations, the one generated by the entry 8 the one generated by 6 and the biggest one which is generated by 5.

The relation between $\gamma_\pi(\mathcal{K})$ and the size of the biggest sequence of adjacent points on the line U, denoted by γ_π^U, is given by

$$\begin{aligned} \gamma_\pi(\mathcal{K}) =& \gamma_\pi^U, \text{ if } \pi \notin Av(12) \\ \gamma_\pi(\mathcal{K}) =& n, \text{ if } \pi \in Av_n(12) \\ \gamma_\pi^U =& 0, \text{ if } \pi \in Av(12) \end{aligned}$$

It follows that we can easily relate the number of permutations having $\gamma_\pi(\mathcal{K}) = j$ with the number of permutations with $\gamma_\pi^U = j$. The number of π's such that γ_π^U is *at most* j is the subject of study in this section.

Given π, it is useful to consider the point $d(\pi)$ defined as the rightmost point placed on the line D. With the terminology of Section 3.1.2, the parameter $v = v(\pi)$ is defined as the number of points placed on the line U which are covering $d(\pi)$. In the permutation of Fig. 3 we have $v(\pi) = 0$. Looking at the recursive contruction of $Av(\mu)$, which has been defined in Section 3.1.2, we observe that, in order to create only permutations satisfying $\gamma^U \leq j$, it is sufficient to avoid at each step the construction of a permutation with a v-value greater than j. This corresponds to use the rules of the form

$$(l) \rightsquigarrow (l') \text{ (with } l' \leq l) \tag{14}$$

no more than j times consecutively.

If we put the same restrictions on a particular recursive construction holding for *Dyck* paths, it turns out that the statistic $\gamma_\pi^U \leq j$ is equivalent to determine the number of those paths avoiding the pattern $U^{j+2}D$ (defined below) and having a fixed size. This problem on paths has been deeply studied in the literature, see for example [8] as well as the following sequences of [9]: $A001006$, i.e., the one of *Motzkin* numbers, for the case $j = 1$, $A036765$ for the case $j = 2$ and $A036766$ for $j = 3$.

We assume the reader is familiar with the definition of Dyck paths. The set of Dyck paths of semi-length n is denoted by $\mathcal{D}_n$ and it is a well known fact that $|\mathcal{D}_n| = c_n$. We represent a path p as a sequence of up U and down D steps. We define a *block* of p as a minimal sub-string (made of consecutive entries) which is still Dyck path and which starts at height 0 in p. For example, if $p = UUDUDDUD$, then there are only two blocks inside p. More precisely, we find $UUDUDD$ which is made by the first six steps and UD which corresponds to the last two entries of p.

The set $\mathcal{D}_{n+1}$ can be generated recursively by the paths in $\mathcal{D}_n$ according to a construction which we use to show the main result of this section. Let

$$p = b_1 \ldots b_{k-1} b_k$$

be a path of $\mathcal{D}_n$ decomposed in terms of its blocks, where $k > 0$. In order to create a path p' of semi-length $n+1$ we add two steps U and D inside p in $k+1$ possible ways. Obtaining

$$p' = UDp \text{ or} \tag{15}$$

$$p' = Ub_1 \ldots b_i D b_{i+1} \ldots b_k \quad (1 \leq i \leq k). \tag{16}$$

In both cases we add an U step at the beginning creating the leftmost block of p'. If p is a path, let $l(p)$ be the number of its blocks.

The construction above corresponds then to the generating tree associated with the rules already described in (12). In particular, (15) is associated with $(l) \rightsquigarrow (l+1)$.

Now, we are ready to prove the following result.

Proposition 5 *The number of permutations $\pi \in Av_n(\mu)$ having $\gamma_\pi^U \leq j$ equals the number of paths in $\mathcal{D}_n$ avoiding $U^{j+2}D$*

Proof. Considering what we have already shown previously, we have to prove two facts: *i*) if we use the rule $(l) \rightsquigarrow (l')$ with $l' \leq l$ for more than j consecutive times in the construction of a path p, then p contains the considered pattern; *ii*) if p contains $U^{j+2}D$, then it has been created using rule (14) consecutively at least $j+1$ times. To prove *i*) it is enough to observe that we obtain a path which starts as $U^{j+1}b_1...$ if we start from a generic path $b_1...b_k$ and we apply (14) $j+1$ times consecutively. The claim follows because b_1 starts as $b_1 = U^iD...$, with $i > 0$. To prove *ii*) let us suppose that $U^{j+2}D$ is inside a path p and let us denote by T the generating tree associated with rules (15) and (16). Observe that we can move inside T going from p to the root simply removing, step-by-step, the left-most entry U of p and the corresponding D. We must find, at some point, an ancestor of p which starts as $U^{j+2}D....$ To buid this ancestor the last $j+1$ applications of the construction rule belong to case (16). □

3.2 $\mu = 1-32$ and right-left avoiding trees

In this section we consider the permutations avoiding the generalized patterns $1-32$. According to [1] a permutation π contains the pattern $1-32$ when there are three indices $j_1 < j_2 < j_3$ such that $j_3 = j_2+1$ and upon rescaling $\pi_{j_1}\pi_{j_2}\pi_{j_3}$ one obtains the pattern 132.

The mapping ϕ described in Section 2 gives a nice representation of these permutations in terms of the associated trees. A tree $t \in \mathcal{T}_n$ is said to be a *right-left avoiding* if all nodes of t can be reached starting from the root of t with a sequence of steps which does not contain a left step after a right one. We denote by $\mathcal{L}$ the set of right-left avoiding trees. An example of a tree in $\mathcal{L}_{10}$ is depicted in Fig. 5 (a) where the associated permutation is

$$\pi = (10\,8\,5\,6\,9\,4\,7\,2\,1\,3).$$

Observe that from π we can extract the pattern 132 from the entries $6, 9$, and 7 but 9 and 7 are not adjacent in π. Indeed, we can prove

Proposition 6 *A tree $t \in \mathcal{T}$ is right-left avoiding if and only if $\phi(t) \in Av(1-32)$.*

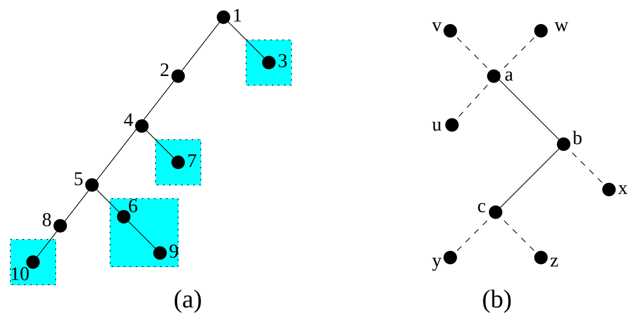

Figure 5: (a) A tree in $\mathcal{L}_{10}$, right oriented chain sub-trees are highlighted; (b) all the possible situations which can produce a right (from a to b)-left (from b to c) sequence of steps in a tree.

Proof. We start showing that if $\phi(t) \in Av(1-32)$, then t is right-left avoiding. In Fig. 5 (b) we have sketched all the possible situations giving a tree which is not right-left avoiding. Indeed, in order to reach c from a, we have to perform a right-left change of direction. The corresponding pattern in the associated permutation is then $v\,u\,a\,y\,c\,z\,b\,x\,w$ (where v and w cannot appear at the same time). Observe that we find the entries a, b and c giving a pattern $1-32$ except when the entry z exists. But, in this case, we can always find a node of the sub-tree generated by z which gives, together with a, b, a pattern $1-32$.

Vice versa, let us take t in $\mathcal{L}$ and let us suppose that there are three entries of $\phi(t)$, i.e., π_{j_1}, π_{j_2} and π_{j_2+1} with $j_1 < j_2$ giving the pattern $1-32$. Observe then that the most recent common ancestor of π_{j_2} and π_{j_2+1} in the tree t must be π_{j_2+1}. Furthermore, looking at the left and right orientation of the nodes of t, we see that π_{j_2} belongs to the left sub-tree of π_{j_2+1}. Thus, we must perform a left step to reach π_{j_2} from π_{j_2+1}. This implies that the entire path from the root of t to π_{j_2+1} is made only of left steps. Thus, there is no place for the entry π_{j_1}. □

$\mathcal{K} = Av(12)$, towards Bell numbers

Let $\pi \in Av_n(\mu)$. In this section the parameter $\gamma_\pi(\mathcal{K})$ corresponds to the size of the biggest chain sub-tree of $\phi^{-1}(\pi)$ which is right oriented. Right oriented chain sub-trees are highlighted in Fig. 5 (a) and, if π is the permutation associated with the right-left avoiding tree depicted there, then $\gamma_\pi(\mathcal{K}) = 2$. In what follows we fix $j \geq 1$ and we determine the value of $l_{j,n}$ defined as the number of trees in $\mathcal{L}_n$ having no right chain sub-tree bigger than j.

In order to determine the numbers $l_{j,n}$ one can proceed using a

recursive construction. A tree t in $\mathcal{L}_j$ can be decomposed according to the following cases

i) t is the empty tree;

ii) t is a right chain tree and $2 \le |t| \le j$;

iii) t is obtained appending two trees to a common root. On the right we attach a right chain tree t' with $1 \le |t'| \le j$ while on the left we put a non-empty tree of $\mathcal{L}_j$;

iv) t is a root to which we append on the left a tree of $\mathcal{L}_j$.

From the previous decomposition follows that the exponential generating function

$$L_j = L_j(x) = \sum_{n\ge 0} \frac{l_{j,n}x^n}{n!}$$

satisfies

$$\begin{aligned} L_j =& 1 + \overbrace{\left(\frac{x^2}{2} + ... + \frac{x^j}{j!}\right)}^{ii)} + \overbrace{\int L_j dx}^{iv)} \\ & \underbrace{+ \sum_{n>0} \frac{x^{n+2}}{(n+2)!}\binom{n+1}{1} + ... + \sum_{n>0} \frac{x^{n+j+1}}{(n+j+1)!}\binom{n+j}{j}}_{iii)} \end{aligned}$$

from which we obtain the derivative

$$L'_j = L_j\left(1 + x + \frac{x^2}{2} + ... + \frac{x^j}{j!}\right) - \frac{x^j}{j!},\ L_j(0) = 1. \tag{17}$$

Solving (17) we find that

$$L_j(x) = E_j(x) + E_j(x)\int_1^0 \frac{z^j}{j!\,E_j(z)}dz - E_j(x)\int_1^x \frac{z^j}{j!\,E_j(z)}dz, \tag{18}$$

where

$$E_j(x) = e^{x+\frac{x^2}{2}+...+\frac{x^{j+1}}{(j+1)!}}.$$

Now recall that we are interested in the coefficients

$$l_{j,n} = L_j^{(n)}(0)$$

and observe that

$$\left[E_j(x)\int_1^x \frac{z^j}{j!\,E_j(z)}dz\right]^{(n)}(x) = E_j^{(n)}(x)\int_1^x \frac{z^j}{j!\,E_j(z)}dz + \sum_{i=0}^{n-1}\left[x^j\left(\frac{E_j^{(n-1-i)}(x)}{j!\,E_j(x)}\right)\right]^{(i)}(x).$$

Then, from (18), we have

$$l_{j,n} = E_j^{(n)}(0) - \frac{1}{j!}\sum_{i=0}^{n-1}\left[x^j\left(\frac{E_j^{(n-1-i)}(x)}{E_j(x)}\right)\right]^{(i)}(0). \qquad (19)$$

The following table shows the values of $l_{j,n}$ for $j = 1, 2, 3$ and $1 \leq n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$j = 1$	1	1	3	6	18	48	156	492	1740	6168
$j = 2$	1	2	4	13	41	146	587	2470	11254	54616
$j = 3$	1	2	5	14	50	190	800	3670	18190	95980

Formula (19) highlights the connection which holds between the sequence (of sequences) $(l_{j,n})_j$ and a family of interesting generating functions, namely $(E_j)_j$. The function E_j appears in the literature several times. For example, in [4], it is related to the problem of counting the number of permutations belonging to $Av_n(12-3, j...21-(j+1))$. See also sequences $A000085$ ($j = 1$), $A001680$ ($j = 2$) and $A001681$ ($j = 3$) of [9]. Furthermore, we can use (19) to re-derive [3], by a limit approach, the enumeration of $Av_n(\mu)$. Indeed, $|Av_n(\mu)|$ corresponds to (19) if $j = \infty$. If the parameter j is not bounded, the presence of the factor x^j in the second summand of (19) implies that, when we plug in $x = 0$, the only contribute which remains is given by $E_\infty^{(n)}(0)$. This is the n-th coefficient of the exponential generating function of the so-called *Bell* numbers which is in fact equal to

$$B(x) = \frac{1}{e}e^{e^x}.$$

Bell numbers ($(b_n)_n$ in what follows) are listed as sequence $A0001100$ of [9] and, as the reader can easily check in the mentioned reference, they give an answer to several enumerative questions.

Unfortunately it is not so easy to cope with (19) if we want to compute the values $l_{j,n}$for large n. Starting from the fact that $n \leq j$ implies $l_{j,n} = |Av_n(\mu)| = b_n$ and looking at the decomposition of right-left avoiding trees we have used to derive (17), we can define a more immediate recursion giving, row by row, the numbers contained in the previous table. Indeed we have

Proposition 7 *For all $n > j+1$, the numbers $l_{j,n}$ can be computed recursively as*

$$l_{j,n} = \sum_{i=0}^{j} \binom{n-1}{i} l_{j,n-i-1}, \tag{20}$$

with initial conditions given by $l_{j,1} = b_1, l_{j,2} = b_2, ..., l_{j,j} = b_j$ and $l_{j,j+1} = b_{j+1} - 1$ (b_n is the n-th Bell number).

Proof. Concerning the initial conditions of the recurrence, we just observe that among the right-left avoiding trees of size $j+1$ there is only one tree which must be excluded to compute $l_{j,j+1}$, that is the right chain tree of size $j+1$. To obtain (20) note that if $n > j+1$, then every tree which has to be counted in $l_{j,n}$ is obtained appending a tree counted in $l_{j,n-i-1}$ as a left sub-tree of a right chain of size $i+1$, where i ranges in $[0, j]$. $\square$

4 Further work: pattern avoidance looking at sub-permutations

Using the terminology of Section 2 it is obvious that, for any pattern σ, each permutation π satisfies the following equivalence

$$\pi \in Av(\sigma) \Longleftrightarrow g_\pi(1) \in Av(\sigma).$$

This is due to the fact that $g_\pi(1) = \pi$. This does in general not hold for $k > 1$. Given $k > 1$ and a pattern σ, it seems interesting to consider permutations π for which

$$\pi \in Av(\sigma) \Longleftrightarrow g_\pi(k) \in Av(\sigma).$$

In other words we seek π such that if it contains σ then also $g_\pi(k)$ contains σ. We denote the set of such permutations by $Av(\sigma; k)$. Furthermore note that in terms of probability one has

$$P(\pi \in Av(\sigma) | g_\pi(k) \in Av(\sigma)) = P(\pi \in Av(\sigma; k))$$

which quantifies the presence of the pattern σ in π depending on its presence in the sub-permutation generated by the entry k.

As an introductory example we consider the case $Av(213; 2)$.

The cardinality of $Av_n(213; 2)$

We proceed counting those permutations $\pi \in \mathcal{S}_n$ such that the pattern 213 is in π and $g_\pi(2) \in Av(213)$. Given π of size n, we denote by m the lowest entry of π such that $m \neq 1$ and $m \notin s_\pi(2)$. Observe that m could not exist. In order to have $\pi \notin Av_n(213; 2)$ only three distinct situations are possible.

i) The entry 2 is placed on the left of 1 and m exists;

ii) the entry 2 is placed on the right of 1 and $g_\pi(m)$ contains 213;

iii) the entry 2 is placed on the right of 1 and $g_\pi(m) \in Av(213)$. Observe that in this case we find 213 in π if and only if $m < M$, where M is defined as the biggest entry of π belonging to $g_\pi(2)$.

Let us denote by i the cardinality of $g_\pi(2)$, then we have $1 \leq i \leq n-1$ and $n-i-1$ gives the number of entries in $g_\pi(m)$. The number of permutations corresponding to *i)* is then

$$\sum_{i=1}^{n-2} c_i \, (n-i-1)! \binom{n-2}{i-1}. \tag{21}$$

Similarly, the number of possible instances of *ii)* is given by

$$\sum_{i=1}^{n-4} c_i \left((n-i-1)! - c_{n-i-1}\right) \binom{n-2}{i-1}. \tag{22}$$

Finally, in case *iii)* we have

$$\sum_{i=1}^{n-2} c_i \, c_{n-i-1} \left(\binom{n-2}{i-1} - 1 \right). \tag{23}$$

Putting together these three results gives the following

Proposition 8 *The number of permutations of size n wich are not in $Av_n(213;2)$ is given by*

$$\begin{aligned} |\mathcal{S}_n \setminus Av(213;2)| = 2(n-2)! \left(\sum_{i=1}^{n-4} \frac{c_i}{(i-1)!} \right) + 2(n-2)(n-3)c_{n-3} \\ + 2(n-2)c_{n-2} - c_n + 2c_{n-1}, \end{aligned} \tag{24}$$

where c_n is the n-th Catalan number.

It follows that the cardinality of $Av_n(213;2)$ is given by

$$|Av_n(213;2)| = n! - |\mathcal{S}_n \setminus Av(213;2)|$$

and, for $3 \leq n \leq 10$, the numbers are

$$5, 16, 68, 392, 2905, 25508, 251188, 2703440,$$

which are not listed in [9].

When n is large we can approximate the sum in parentheses in (24) as a constant

$$k = \sum_{i=1}^{n-4} \frac{c_i}{(i-1)!} = 11.75330... \ . \tag{25}$$

An asymptotic approximation of $|\mathcal{S}_n \setminus Av(213; 2)|$ is then given by

$$2(n-2)!k.$$

The probability that a permutation of size $n \to \infty$ does not belong to $Av(213; 2)$ is then

$$\frac{2k}{n^2}. \tag{26}$$

Asymptotic behaviour of $P(\pi \notin Av_n(\sigma; 2))$

Observe that equation (26) depends only on the constant k (besides n). This constant can be determined with high accuracy from the first few, say 50, summands in (25). This suggests that often one does not need to know for every n the number of permutations in $Av_n(\sigma)$ to determine the behaviour of $Av_n(\sigma; k)/n!$. In fact, if $k = 2$ and for n large,

$$P(\pi \notin Av_n(\sigma; 2)) = \frac{2k_\sigma}{n^2}, \tag{27}$$

where

$$k_\sigma = \sum_{i=1}^{n-1-|\sigma|} \frac{|Av_i(\sigma)|}{(i-1)!}$$

is a constant which can be approximated using the first i' values of $(|Av_n(\sigma)|)_n$, where i' (the upper summation bound) depends on σ. The reasoning which leads to (27) is the following. As done for the case $Av(213; 2)$, let us consider a permutation $\pi \notin Av_n(\sigma; 2)$. Let i be the size of $g_\pi(2)$ and m the lowest entry of π with $m \neq 1$ and $m \notin s_\pi(2)$. There are two basic cases for π depending on the presence of the pattern σ in $g_\pi(m)$. If $g_\pi(m) \notin Av(\sigma)$, the number of possible π's is given by

$$2(n-2)! \left(\sum_{i=1}^{n-1-|\sigma|} \frac{|Av_i(\sigma)|}{(i-1)!} - \sum_{i=1}^{n-1-|\sigma|} \frac{|Av_i(\sigma)|}{(i-1)!} \times \frac{|Av_{n-i-1}(\sigma)|}{(n-i-1)!} \right).$$

If $g_\pi(m) \in Av(\sigma)$ then the number of possible π's is bounded from the top by

$$2(n-2)! \sum_{i=1}^{n-1} \frac{|Av_i(\sigma)|}{(i-1)!} \times \frac{|Av_{n-i-1}(\sigma)|}{(n-i-1)!}.$$

It follows that

$$2(n-2)!\left(k_\sigma - \sum_{i=1}^{n-1-|\sigma|} \frac{|Av_i(\sigma)|}{(i-1)!} \times \frac{|Av_{n-i-1}(\sigma)|}{(n-i-1)!}\right) \tag{28}$$

$$\leq |\mathcal{S}_n \setminus Av(\sigma;2)|$$

$$\leq 2(n-2)!\left(k_\sigma + \sum_{i=n-|\sigma|}^{n-1} \frac{|Av_i(\sigma)|}{(i-1)!} \times \frac{|Av_{n-i-1}(\sigma)|}{(n-i-1)!}\right). \tag{29}$$

The sums appearing in (28) and (29) go to zero for n large and dividing by $n!$ we obtain the desired result

$$\frac{|\mathcal{S}_n \setminus Av(\sigma;2)|}{n!} \sim \frac{2k_\sigma}{n^2}, \text{ with } n \to \infty.$$

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